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A REFINED THEORY OF LAMINATED SHALLOW SHELLS

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Abstract—A new refined but simple shear deformation theory of elastic shells is developed for shells laminated of orthotropic layers. To evaluate the new displacement field assumed which is justified in plates from the three-dimensional elasticity theory, classic types of shallow shells are considered. The boundary value problem is formulated by making use of the principle of virtual power in conjunction with the assumed consistent displacement field. The theory accounts for cosine distribution of the transverse shear strains through thickness of the shell and tangential stress-free boundary conditions on the boundary surfaces of the shell. The theory also accounts for in-plane inertia and rotatory inertia. The Navier type exact solutions are presented in statics and in vibrations for cylindrical and spherical shells under simply supported edge boundary conditions. The theory is of the same order of complexity as the shear deformation theory but is very much more efficient without needing shear correction factors. Some numerical comparisons with other works are made.

1. INTRODUCTION

The more general structural elements are any curved structures which extend continuously to a much greater extent in two dimensions (i.e. on a curved surface) than in the third dimension (i.e. the thickness direction). The structural elements are very useful in engineering design, *especially for the sizing of structures and for developing finite elements*. A shell may have any of a great variety of geometric configurations. Practical shell structures often have various kinds of discontinuities, such as holes, bosses, changes in thickness and stiffeners. Usually, composite-material shells are laminates of many plies or layers.

Because of difficulties involved in deriving two-dimensional theories of shells from three-dimensional equations of elasticity, assumptions of one kind or another must be introduced in the derivation. So, approximate bidimensional linear theories for shells have been developed by making use of an assumed displacement field in powers of the thickness coordinate and a variational theorem. An integration with respect to the thickness coordinate supplies the governing differential equations and consistent boundary conditions in terms of unknown generalized coordinates which are independent of the thickness coordinate. An asymptotic integration of the elasticity equations has been employed for isotropic shells, Goldenveizer (1963) and for nonhomogeneous shells Widera and Logan (1980). To derive two-dimensional theories from three-dimensional equations, a method has been presented by Cheng (1977) by expanding solutions in Taylor series.

Surveys of various classical shell theories can be found in the works of Naghdi (1971), Bert and Francis (1974) and Bert (1980). Classical shell theories were developed originally for thin elastic shells, based on the Kirchhoff–Love plate's assumptions and various degrees of approximation on the curvatures, except the Langhaar and Boresi (1958) theory which is exact in terms of the Kirchhoff–Love hypothesis. These classical shell theories are those of Donnell (1933), Morley (1959), Love's first approximation, Love (1927), Sanders (1959), Novozhilov (1964), Love's second approximation, and Flügge (1960). For a more detailed discussion of these various shell theories, the reader is referred to Naghdi (1971).

From Koiter (1959), refinements to Love's first approximation theory of thin elastic shells are not sufficient, except if the effects of transverse shear and normal stresses are taken into account in the refined theory. Then, the transverse normal stress is of the order: thickness to radius of curvature ratio times the bending stresses, whereas the transverse shear stresses deduced from equilibrium equations are of the order: thickness to length along the side of the shell times the bending stresses, Reddy and Liu (1985).

The effects of transverse shear and normal stresses in shells were considered by Reissner (1952) for shells of revolution, Naghdi (1957) to arbitrarily doubly curved shells and Dong and Tso (1972). A very good synthesis in dynamics has been made by Greenspon (1960) for homogeneous shells.

Higher-order shell theories in which a displacement field of polynomial form, a degree greater than one is assumed, have been developed for cylindrical shells by Whitney and Sun (1974) and by Bhimaraddi (1984), for doubly curved shallow shells by Reddy and Liu (1985), and for general shells by Doxsee (1989) for the purpose of removing the inaccuracies in the laminated shells of the shear deformation theory which accounts only for constant transverse shear stresses through thickness. In addition, the shear correction factors which the shear deformation theory needs, are not consistent.

In this paper, *a new type of approach is proposed* for developing a simple and refined shear-deformation theory for moderately thick laminated shells. The theory contains the same independent generalized displacements as in the shear deformation theory, and is based on a new assumed displacement field in which the shear is represented by a sine function. This means is justified from a three-dimensional point of view in plates and allows us to have a cosine distribution of transverse shear stresses through the thickness of the shell. Also, unlike some of the shear deformation theories, the present analysis does not involve the determination of any unknown shear coefficients. The model is evaluated by comparing the results obtained from it, as well as from shear deformation theory Reddy (1984a), and higher-order shear deformation theories (Reddy and Liu, 1985; Bhimaraddi, 1984), with, when possible, the (unfortunately rare) exact three-dimensional results. Numerical results are obtained for laminated spherical domes, laminated and isotropic short cylindrical shells in static states and in vibration. The objective of this research is to develop efficient (i.e. simple and accurate) tools for the design and the sizing of structures, in linear and nonlinear behaviour, and if necessary for structures made of composite materials. So, a finite element approximation may be constructed using an efficient theoretical model.

2. THE NEW BASIC TWO-DIMENSIONAL MODEL

In previous works on structural mechanics, and wave propagation in bars and beams, it has been suggested that trigonometric functions be used in the kinematics (Touratier, 1980, 1987) in place of polynomial developments of the transverse coordinates. Recently (Touratier, 1989), it was proposed that this idea be extended to the plate theory. Some numerical results for: a simply-supported laminated plate under a doubly sinusoidal normal static pressure; a simply-supported sandwich plate under a uniform normal static pressure; fundamental, free vibration mode of a simply-supported sandwich square plate, have been carried out in comparison with three-dimensional exact solutions. The comparisons have shown the efficiency of the proposed model through the accuracy of the numerical results and the simplicity of the theory. A similar approach was used by Stein (1986) for plates, see the review article by Reddy (1990).

To explain the basic model, we start with plates under a normal loading such as pressure, concentrated load for instance. Let us consider a body occupying the domain

$$\Omega = A \times [-h/2 \leqslant z \leqslant h/2]$$

in a Cartesian coordinate system $(x_1, x_2, x_3 = z)$, with A as an arbitrary region in the (x_1, x_2) plane, and with diameter $(A) \gg h$, the thickness of the body Ω. To have only the five classic independent generalized displacements, to immediately satisfy zero transverse shear stress conditions on the top and bottom surfaces of the plate, and to have a higher order and simple kinematic, we propose to write the variation of displacements (U_1, U_2, U_3) through the thickness of the plate in the following form:

$$U_\alpha(x_1,x_2,z,t) = u_\alpha(x_1,x_2,t) - zw_{,\alpha}(x_1,x_2,t) + (h/\pi)\sin(\pi z/h)\gamma_\alpha^0(x_1,x_2,t)$$

$$U_3(x_1,x_2,z,t) = w(x_1,x_2,t), \quad \alpha = 1 \quad \text{or} \quad 2, \quad w_{,\alpha} = \partial w/\partial x_\alpha, \tag{1}$$

where t is time, (u_α, w) are the displacements of a point in the middle of plane A and γ_α^0 are the transverse shear strains at $z = 0$. The displacement field (1) implies that the transverse shear strains are zero on surfaces $z = \pm h/2$, and are functions of the even kind of the thickness coordinate z, which is consistent. Because the sine function has an infinite radius of convergence, from eqn (1) we can write the in-plane displacements:

$$U_\alpha = u_\alpha - zw_{,\alpha} + \frac{h}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{(2p+1)!}\left(\frac{\pi z}{h}\right)^{2p+1}\gamma_\alpha^0, \quad p \in \mathbb{N}$$

$$U_\alpha = u_\alpha - zw_{,\alpha} + z\gamma_\alpha^0 - \frac{\pi^2}{3!h^2}z^3\gamma_\alpha^0 + \frac{\pi^4}{5!h^4}z^5\gamma_\alpha^0 - \frac{\pi^6}{7!h^6}z^7\gamma_\alpha^0 + \cdots. \tag{2}$$

Since $\gamma_\alpha^0 = \omega_\alpha + w_{,\alpha}$ where ω_α are the rotations at $z = 0$ of normals to the midplane A with respect to the x_α axes, we note:

(1) if $\gamma_\alpha^0 = 0$ we obtain the Kirchhoff–Love theory,
(2) if we develop the first order sine function, we obtain the Mindlin theory,
(3) if we develop the sine function to the third order only, then the in-plane kinematic (2) is

$$U_\alpha = u_\alpha - zw_{,\alpha} + z\gamma_\alpha^0 - \frac{\pi^2}{6h^2}z^3\gamma_\alpha^0 + \cdots$$

which is of the same order as the Levinson (1980) and Reddy (1984b) kinematic. In fact, the Levinson and Reddy in-plane kinematics is given by

$$U_\alpha = u_\alpha - z\left[\omega_\alpha - \frac{4}{3}\left(\frac{z}{h}\right)^2(\omega_\alpha - w_{,\alpha})\right] = u_\alpha - zw_{,\alpha} + z\gamma_\alpha^0 - \frac{4}{3h^2}z^3\gamma_\alpha^0.$$

In the Stein theory, eqn (1) is taken under the following form [which is not equivalent to eqn (1)]:

$$U_\alpha = u_\alpha + \frac{z}{h}u_\alpha^a + \sin\frac{\pi z}{h}u_\alpha^s;$$

$$U_3 = w + \cos\frac{\pi z}{h}w^c.$$

So, with the Stein theory, boundary conditions are not satisfied for the shear stresses on the top and bottom surfaces of the plate, and the theory involves eight independent generalized displacements.

To develop our theory, we will, of course, keep the sine function intact. In fact, the kinematic proposed in (1) can be justified from the three-dimensional point of view by using the excellent work of Cheng (1979). Cheng has presented a method for the solution of three-dimensional elasticity equations for the problem of thick plates. Through this method three governing differential equations, the well-known biharmonic equation $\nabla^2\nabla^2 w = -q/D$ (∇^2 is the Laplacian, D the bending rigidity and q is the transverse load), a shear equation $(\nabla^2 - (2p+1)^2\pi^2/h^2)s(x_1,x_2) = 0$ (s is a shear function) and a transcendental equation $(1/\nabla^2)(1-\sin(h\nabla)/h\nabla)H(x_1,x_2) = 0$ (H is a stress function) are deduced directly and systematically from Navier's equations. Only the third equation involves a transverse normal stress without shear, but the solution contains higher-order derivatives. The second equation is called the shear equation because its solution is related to the pure shear deformation in the bending of thick plates. So, from Cheng (1979), the solution of the shear equation is such that $U_1 = \sin((2p+1)\pi z/h)x_2$; $U_2 = -\sin((2p+1)\pi z/h)x_1$ and $U_3 = 0$. Hence the shear in our theory can be deduced from the above solution by imposing $p = 0$ and also $(h/\pi)\gamma_1^0 = x_2$; $(h/\pi)\gamma_2^0 = -x_1$; eqn (3). Then, the modelling of the shear in (1) is consistent. In addition, keeping functions γ_α^0 in eqn (1) allows us to find the Mindlin's theory by developing the sine of the first order and to facilitate a finite element approximation. Other terms in (1) are membrane displacements with u_α, and Kirchhoff–Love bending w which is equivalent to the first term of the asymptotic expansion in the three-dimensional elasticity equations. The bending beam kinematics in plane (x_2, x_3) can be deduced from (1) by imposing $u_\alpha = 0$ and $\gamma_1^0 = 0$, γ_2^0 and w being function of x_2 and time.

Now we are going to build the shell model following a procedure similar to that presented above for plates. A shell of constant thickness h is considered, and the points of the shell and its boundary are denoted by Ω and Γ, respectively. The boundary of the shell is the union of the upper surface Γ_{upper}, the lower surface Γ_{lower} and the edge faces Γ_{edge}. The set of points lying halfway between the upper and lower surfaces is called the midsurface and is denoted by A. The outward unit vector normal to Γ is denoted $\mathbf{n}$ and the intersection of A and Γ_{edge} is denoted $\mathscr{C}$. In order to obtain numerical solutions to the governing equations that we are to derive, it is necessary to express the equations in component form. Let (ξ_1, ξ_2, ζ) denote the orthogonal principal-curvilinear coordinates (or shell coordinates) such that the ξ_1- and ξ_2-curves are lines of curvature on the midsurface $\zeta = 0$, ζ-curves are straight lines perpendicular to the surface $\zeta = 0$. The values of the principal radii of curvature of the middle surface are denoted by R_1 and R_2. Then, curves of constant x_1 coincide with curves of principal curvature $1/R_2$ of the midsurface, and curves of constant x_2 coincide with curves of principal curvature $1/R_1$. The distance $\mathrm{d}s$ between points $\mathbf{p} = (\xi_1, \xi_2, 0)$ and $\mathbf{p}' = (\xi_1 + \mathrm{d}\xi_1, \xi_2 + \mathrm{d}\xi_2, 0)$ on the midsurface is determined by

$$(\mathrm{d}s)^2 = \alpha_1^2(\mathrm{d}\xi_1)^2 + \alpha_2^2(\mathrm{d}\xi_2)^2 \tag{4}$$

where α_1 and α_2 are the surface metrics such that

$$\alpha_i^2 = (\partial\mathbf{p}/\partial\xi_i)(\partial\mathbf{p}/\partial\xi_i), \quad i = 1 \quad \text{or} \quad 2. \tag{5}$$

In this expression, $\mathbf{p}(\xi_1, \xi_2, 0)$ is a point of the surface A of the shell, α_1 and α_2 are scalars which are functions of position $(\xi_1, \xi_2, 0)$ on the midsurface. The four quantities α_1, α_2, R_1, R_2 define the shape of the shell and are not independent. The distance $\mathrm{d}S$ between points $\mathbf{P} = (\xi_1, \xi_2, \zeta)$ and $\mathbf{P}' = (\xi_1 + \mathrm{d}\xi_1, \xi_2 + \mathrm{d}\xi_2, \zeta + \mathrm{d}\zeta)$ out of the midsurface is given by

$$(\mathrm{d}S)^2 = L_1^2(\mathrm{d}\xi_1)^2 + L_2^2(\mathrm{d}\xi_2)^2 + L_3^2(\mathrm{d}\zeta)^2 \tag{6}$$

where L_1, L_2 and L_3 are the Lamé coefficients,

$$L_1 = \alpha_1\left(1 + \frac{\zeta}{R_1}\right), \quad L_2 = \alpha_2\left(1 + \frac{\zeta}{R_2}\right), \quad L_3 = 1. \tag{7}$$

So, the point $\mathbf{p}$ of the midsurface closest to $\mathbf{P}$ is related to $\mathbf{P}$ via $\mathbf{P} = \mathbf{p} + \zeta\mathbf{n}'(\mathbf{p})$ where ζ is the distance between points $\mathbf{p}$ and $\mathbf{P}$, and $\mathbf{n}'(\mathbf{p})$ is the unit vector normal to the midsurface at $\mathbf{p}$. Then the displacement field is taken under the following form, from considerations regarding the above plate model:

$$\begin{aligned}
\bar{U}_\beta(\xi_1, \xi_2, \zeta, t) &= \frac{L_\beta}{\alpha_\beta}\bar{u}_\beta(\xi_1, \xi_2, t) - \frac{\zeta\bar{w}_{,\beta}}{\alpha_\beta}(\xi_1, \xi_2, t) + \frac{h}{\pi}\sin\frac{\pi\zeta}{h}\bar{\gamma}_\beta^0(\xi_1, \xi_2, t) \\
\bar{U}_3(\xi_1, \xi_2, \zeta, t) &= \bar{w}(\xi_1, \xi_2, t), \quad \bar{w}_{,\beta}(\xi_1, \xi_2, t) = \partial\bar{w}/\partial\xi_\beta, \quad \beta = 1 \quad \text{or} \quad 2.
\end{aligned} \tag{8}$$

In eqns (8), $(\bar{U}_1, \bar{U}_2, \bar{U}_3)$ are the displacements along the (ξ_1, ξ_2, ζ) coordinates; $(\bar{u}_1, \bar{u}_2, \bar{w})$

are the displacements of a point on the middle surface and $\bar{\gamma}_1^0$ and $\bar{\gamma}_2^0$ are the shear strains at $\zeta = 0$. This choice in eqn (8) is dictated by the same requirements as those in plates so as to have an efficient model (i.e. simplicity and accuracy of the model), only five independent generalized displacements, higher-order shear deformation to avoid shear correction factors, and zero transverse shear stress on the upper and lower surfaces of the shell. In fact, for general shells, shear strains are defined by [see Doxsee (1989)]:

$$2\varepsilon_{\beta\zeta} = \frac{\alpha_\beta}{L_\beta}\left(\frac{1}{\alpha_\beta}\frac{\partial \bar{U}_3}{\partial \xi_\beta} - \frac{\bar{U}_\beta}{R_\beta}\right) + \frac{\partial \bar{U}_\beta}{\partial \zeta}.$$

With the present theory, from eqn (8) we have

$$2\varepsilon_{\beta\zeta} = \frac{1}{\alpha_\beta}\bar{w}_{,\beta} - \frac{\bar{u}_\beta}{R_\beta} + \frac{\zeta}{\alpha_\beta R_\beta}\bar{w}_{,\beta} - \frac{h}{\pi R_\beta}\sin\frac{\pi\zeta}{h}\bar{\gamma}_\beta^0 + \frac{\bar{u}_\beta}{R_\beta} - \frac{1}{\alpha_\beta}\bar{w}_{|\beta} + \cos\frac{\pi\zeta}{h}\bar{\gamma}_\beta^0.$$

For shallow shells for which $h/R_\beta \ll 1$, the preceding shear strains become

$$2\varepsilon_{\beta\zeta} \cong \cos\frac{\pi\zeta}{h}\bar{\gamma}_\beta^0.$$

Then, zero shear-stress conditions on the upper and lower surfaces of shallow shells are well satisfied, provided the shell material is not more than monoclinic.

3. TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM FOR SHELLS

In this section a simple and refined theory of composite shells is developed. The shell considered has a uniform thickness which is much smaller than the shell's radii of curvature. The shell may be composed of a single material or several different materials bonded together in layers, each layer having a constant thickness. Each layer may be isotropic or orthotropic. The material properties are assumed to be linearly elastic. A consistent combination of displacements (essential boundary conditions), forces and moments (natural boundary conditions) are specified along the edges of the shell. The displacement of each point of the shell is taken to be small compared to the thickness.

The following problem is addressed: given the initial geometry of the shell, its material properties, the prescribed edge forces and displacements; the displacements and stresses at every point of the shell are required. The equilibrium equations and boundary conditions are derived via the principle of virtual power, Germain (1986). Let Ω be a shell with tractions $\mathbf{F}$ prescribed along part of its boundary $\Gamma_\sigma \subset \Gamma_{\text{edge}}$ and displacements prescribed along the other part $\Gamma_u \subset \Gamma_{\text{edge}}$, where the symbol $\subset$ represents a subset. The upper and lower surfaces of the shell are taken to be traction free.

To use the principle of virtual power to derive equilibrium equations and boundary conditions, we start by defining two spaces $\mathcal{U}$ and $\hat{\mathcal{U}}$ such that ($\beta = 1$ or 2):

$$\mathcal{U} = \left\{\bar{U}_\beta = \frac{L_\beta}{\alpha_\beta}\bar{u}_\beta - \frac{\zeta\bar{w}_{|\beta}}{\alpha_\beta} + f(\zeta)\bar{\gamma}_\beta^0,\quad \bar{U}_3 = \bar{w} : (\bar{u}_\beta, \bar{\gamma}_\beta^0) \in H^1(A)\times H^1(A),\right.$$

$$\bar{w}\in H^2(A), \bar{u}_\beta\,\bar{\gamma}_\beta^0 \text{ and } \bar{w} \text{ and } \bar{w}_{|\beta} \text{ specified on } \Gamma_u \subset \Gamma_{\text{edge}}$$

$$\left.\text{(essential boundary conditions)},\quad f(\zeta) = \frac{h}{\pi}\sin\frac{\pi\zeta}{h}\right\}, \tag{9}$$

$$\hat{\mathcal{U}} = \left\{\hat{\bar{U}}_\beta = \frac{L_\beta}{\alpha_\beta}\hat{\bar{u}}_\beta - \frac{\zeta\hat{\bar{w}}_{|\beta}}{\alpha_\beta} + f(\zeta)\hat{\bar{\gamma}}_\beta^0,\quad \hat{\bar{U}}_3 = \hat{\bar{w}} : (\hat{\bar{u}}_\beta, \hat{\bar{\gamma}}_\beta^0) \in H^1(A)\times H^1(A),\right.$$

$$\left.\hat{\bar{w}}\in H^2(A),\ \hat{\bar{U}} \text{ is zero on } \Gamma_u \subset \Gamma_{\text{edge}},\quad f(\zeta) = \frac{h}{\pi}\sin\frac{\pi\zeta}{h}\right\} \tag{10}$$

where $H^1(A)$ is a Sobolev space.

The space $\mathscr{U}$ is the space of admissible displacements $\bar{U}$ defined in eqns (8) and the space $\hat{\mathscr{U}}$ is the space of virtual velocities $\hat{\bar{U}}$ which must be considered at a fixed time. The principle of virtual power states that: find $(\bar{u}_\beta, \bar{w}, \bar{\gamma}^0_\beta) \in \mathscr{U}$ such that for every $(\hat{\bar{u}}_\beta, \hat{\bar{w}}, \hat{\bar{\gamma}}^0_\beta) \in \hat{\mathscr{U}}$, then we have (summation on i and $j = 1, 2, 3$):

$$\int_\Omega \rho \ddot{\bar{U}} \hat{\bar{U}} \,\mathrm{d}V = -\int_\Omega \sigma_{ij} \hat{D}_{ij} \,\mathrm{d}V + \int_\Omega \mathbf{f} \hat{\bar{U}} \,\mathrm{d}V + \int_{\Gamma_\sigma} \mathbf{F} \hat{\bar{U}} \,\mathrm{d}a. \tag{11}$$

To simplify, we shall write (11) in the corresponding form:

$$\hat{\mathscr{P}}_a = \hat{\mathscr{P}}_i + \hat{\mathscr{P}}_d + \hat{\mathscr{P}}_c, \quad \forall \hat{\bar{\mathbf{U}}} \in \hat{\mathscr{U}} \tag{11a}$$

where $\hat{\mathscr{P}}_a$ is the virtual power of the inertial forces, $\hat{\mathscr{P}}_i$ the virtual power of the internal forces due to stresses, $\hat{\mathscr{P}}_d$ the virtual power of the volume forces and $\hat{\mathscr{P}}_c$ the virtual power of the external contact forces.

In eqn (11) σ is the stress tensor, $\hat{D}$ the virtual strain rate tensor, ρ the mass density, $\ddot{\bar{U}}$ the acceleration vector ($\ddot{\bar{U}} = \partial^2 \bar{U}/\partial t^2$), and $\mathbf{f}$ the body forces. Virtual velocity measures, $\hat{\bar{u}}_\alpha, \hat{\bar{w}}, \hat{\bar{\gamma}}^0_\alpha$ have been defined by eqn (10). The tensor $\hat{D}$ is defined by Doxsee (1989):

$$\hat{D} = \tfrac{1}{2}(\tilde{\nabla}^{\mathrm{T}} \hat{\bar{U}} + \tilde{\nabla} \hat{\bar{U}}), \quad \tilde{\nabla} = (\mathbf{1} - \zeta \mathbf{b})^{-1} \nabla + \mathbf{n}' \otimes \frac{\partial}{\partial \zeta} \tag{12}$$

where the superscript T denotes the transpose, $\tilde{\nabla}$ the gradient operator on three-dimensional space, ∇ the gradient operator on the midsurface, $\mathbf{1}$ the identity tensor on the midsurface, $\mathbf{b} = -\nabla \mathbf{n}'$ the curvature tensor of the midsurface, $\mathbf{n}'$ the unit vector normal to the midsurface at $\mathbf{p}$ and $\otimes$ is the tensor product operator.

To obtain numerical results and to evaluate the theory without any other approximation such as finite element approximation of eqn (9) to eqn (11), we restrict the theory to shells such as $h/R_\alpha \ll 1$ and with constant radius of curvature. Then, from (12) and (10) the virtual strain rates are in curvilinear coordinates (no summation on $\beta = 1$ or 2):

$$\begin{aligned}
\hat{D}_{\beta\beta} &= \frac{1}{\alpha_\beta} \hat{\bar{u}}_{\beta|\beta} - \zeta \frac{\hat{\bar{w}}_{|\beta\beta}}{\alpha_\beta^2} + f(\zeta) \frac{\hat{\bar{\gamma}}^0_{\beta|\beta}}{\alpha_\beta} + \frac{\hat{\bar{w}}}{R_\beta} \\
2\hat{D}_{12} &= \frac{1}{\alpha_2} \hat{\bar{u}}_{1|2} + \frac{1}{\alpha_1} \hat{\bar{u}}_{2|1} - 2\zeta \frac{\hat{\bar{w}}_{|12}}{\alpha_1 \alpha_2} + f(\zeta) \left(\frac{\hat{\bar{\gamma}}^0_{1|2}}{\alpha_2} + \frac{\hat{\bar{\gamma}}^0_{2|1}}{\alpha_1} \right) \\
2\hat{D}_{\beta\zeta} &= f_{|\zeta} \hat{\bar{\gamma}}^0_\beta.
\end{aligned} \tag{13}$$

By combining eqns (9)–(13), integrating eqn (11) [therefore (11a)] through the thickness, and performing other algebraic manipulations, one obtains for the terms in (11a):

$$\begin{aligned}
\hat{\mathscr{P}}_a = {} & \int_A \sum_{\beta=1}^{2} \left\{ \left(I_0 + \frac{2}{R_\beta} I_1 + \frac{1}{R_\beta^2} I_2 \right) \ddot{\bar{u}}_\beta - \frac{1}{\alpha_\beta} \left(I_1 + \frac{1}{R_\beta} I_2 \right) \ddot{\bar{w}}_{|\beta} + \left(J_1 + \frac{1}{R_\beta} K \right) \ddot{\bar{\gamma}}^0_\beta \right\} \hat{\bar{u}}_\beta \,\mathrm{d}a \\
& + \int_A \sum_{\beta=1}^{2} \left\{ \left(J_1 + \frac{1}{R_\beta} K \right) \ddot{\bar{u}}_\beta - \frac{K}{\alpha_\beta} \ddot{\bar{w}}_{|\beta} + J_2 \ddot{\bar{\gamma}}^0_\beta \right\} \hat{\bar{\gamma}}^0_\beta \,\mathrm{d}a \\
& + \int_A \left\{ I_0 \ddot{\bar{w}} \hat{\bar{w}} + \sum_{\beta=1}^{2} \left[-\frac{1}{\alpha_\beta} \left(I_1 + \frac{1}{R_\beta} I_2 \right) \ddot{\bar{u}}_\beta + \frac{I_2}{\alpha_\beta^2} \ddot{\bar{w}}_{|\beta} - \frac{K}{\alpha_\beta} \ddot{\bar{\gamma}}^0_\beta \right] \hat{\bar{w}}_{|\beta} \right\} \mathrm{d}a
\end{aligned} \tag{14}$$

$$\hat{\mathscr{P}}_i = -\int_A \sum_{\gamma,\beta=1}^{2} N_{\gamma\beta}\frac{\hat{u}_{\gamma|\beta}}{\alpha_\beta}\,da - \int_A \sum_{\beta=1}^{2} N_{\beta\beta}\frac{\hat{w}}{R_\beta}\,da + \int_A \sum_{\beta,\gamma=1}^{2} M_{\gamma\beta}\frac{\hat{w}_{|\gamma\beta}}{\alpha_\gamma\alpha_\beta}\,da$$

$$-\int_A \sum_{\beta,\gamma=1}^{2} \tilde{M}_{\beta\gamma}\frac{\hat{\gamma}^0_{\gamma|\beta}}{\alpha_\beta}\,da - \int_A \sum_{\beta=1}^{2} \tilde{Q}_{\beta\zeta}\hat{\gamma}^0_\beta\,da \quad (15)$$

$$\hat{\mathscr{P}}_d = \int_A \left(\sum_{\beta=1}^{2} \{p_\beta \hat{u}_\beta + \tilde{m}_\beta \hat{\gamma}^0_\beta\} + q\hat{w}\right) da \quad (16)$$

$$\hat{\mathscr{P}}_c = \int_{\mathscr{C}} \left(\sum_{\beta=1}^{2} \{T_\beta \hat{u}_\beta + C_\beta \hat{\gamma}^0_\beta\} - M_f \hat{w}_{|n} + T_\zeta \hat{w}\right) ds. \quad (17)$$

The new terms appearing in these equations are automatically defined as below ($n = 0, 1, 2$ and $\beta = 1, 2$):

$$I_n = \int_{-h/2}^{+h/2} \rho\zeta^n\, d\zeta, \quad J_\beta = \int_{-h/2}^{+h/2} \rho f^\beta(\zeta)\, d\zeta, \quad K = \int_{-h/2}^{+h/2} \rho\zeta f(\zeta)\, d\zeta. \quad (18)$$

In eqns (14), (15) and (16), the integrals are surface integrals over A and in eqn (17), the integral is a line integral along the intersection of Γ and Γ_σ, which is denoted $\mathscr{C}$. Also appearing in eqn (15) are the stress resultants

$$N_{\gamma\beta} = \int_{-h/2}^{+h/2} \sigma_{\gamma\beta}\, d\zeta, \quad M_{\gamma\beta} = \int_{-h/2}^{+h/2} \zeta\sigma_{\gamma\beta}\, d\zeta$$

$$\tilde{M}_{\gamma\beta} = \int_{-h/2}^{+h/2} f(\zeta)\sigma_{\gamma\beta}\, d\zeta, \quad \tilde{Q}_{\gamma\zeta} = \int_{-h/2}^{+h/2} f_{|\zeta}\sigma_{\gamma\zeta}\, d\zeta \quad (19)$$

where γ and $\beta = 1$ or 2. In eqn (19), $N_{\gamma\beta}$ are the membrane generalized stresses, $M_{\gamma\beta}$ the first flexion and torsion generalized stresses, $\tilde{M}_{\gamma\beta}$ the higher-order flexion and torsion generalized stresses, $\tilde{Q}_{\gamma\zeta}$ the higher-order transverse shear generalized stresses. The body force resultants are defined in eqn (16) from the definition of the virtual power of volume forces $\hat{\mathscr{P}}_d = \int_\Omega \mathbf{f}\,\hat{U}\,dv$ and taking into account (10) to define the virtual velocity $\hat{U}$. The prescribed traction resultants in eqn (17) are defined using the definition of the virtual power of surface forces $\hat{\mathscr{P}}_c = \int_{\Gamma_\sigma} \mathbf{F}\hat{U}\,da$ and the definition of the virtual velocity in eqn (10). Finally, in eqn (17), $\hat{w}_{|n}$ is the normal derivative along the curve $\mathscr{C}$, T_β and T_ζ are forces prescribed along the edge of the shell, C_β and M_f are moments prescribed along the edge of the shell. In eqn (16), q is the classic normal transverse charge to the shell, p_β and $\tilde{m}_\beta$ are respectively surface forces and moments applied inside A.

Now it is sufficient to apply the principle of virtual power to obtain the formulation of the boundary value problem. Then, (11a) with eqns (14)–(17) and by making use of the integration by parts, imply:

—for all $(\hat{u}_\beta, \hat{w}, \hat{\gamma}^0_\beta) \in \hat{\mathscr{U}}$, the equilibrium equations in the midsurface A of the shell Ω:

$$\Gamma^{(u)}_\beta = \sum_{\gamma=1}^{2} \frac{N_{\beta\gamma|\gamma}}{\alpha_\gamma} + p_\beta, \quad \beta = 1 \quad \text{or} \quad 2$$

$$\Gamma^{(w)} = \sum_{\beta,\gamma=1}^{2} \left(\frac{M_{\beta\gamma|\beta\gamma}}{\alpha_\beta\alpha_\gamma} - \frac{N_{\beta\beta}}{R_\beta}\right) + q$$

$$\Gamma^{(\gamma)}_\beta = \sum_{\gamma=1}^{2} \frac{\tilde{M}_{\beta\gamma|\gamma}}{\alpha_\gamma} - \tilde{Q}_{\beta\zeta} + \tilde{m}_\beta, \quad \beta = 1 \quad \text{or} \quad 2; \quad (20)$$

—for all $(\hat{\dot{u}}_\beta, \hat{\dot{w}}, \hat{\dot{w}}_{,n}, \hat{\dot{\gamma}}^0_{\gamma\beta}) \in \hat{\mathscr{U}}(\mathscr{C}) \subset \hat{\mathscr{U}}$, the natural boundary conditions at the edge $\mathscr{C}$ of the midsurface $\mathscr{A}$ of the shell Ω:

$$
\begin{aligned}
-\sum_{\gamma=1}^{2} N_{\beta\gamma} n_\gamma + T_\beta = 0, \quad \beta = 1 \quad \text{or} \quad 2 \\
-\bar{\Gamma}^{(w)} = -\sum_{\beta,\gamma=1}^{2} \frac{(M_{\beta\gamma} n_\gamma m_\beta)_{,\tau}}{\alpha_\gamma} - \sum_{\beta,\gamma=1}^{2} \frac{M_{\beta\gamma,\gamma} n_\beta}{\alpha_\gamma} + T_\zeta \\
\sum_{\beta,\gamma=1}^{2} M_{\beta\gamma} n_\beta n_\gamma - M_\Gamma = 0 \\
-\sum_{\gamma=1}^{2} \tilde{M}_{\beta\gamma} n_\gamma + C_\beta = 0, \quad \beta = 1 \quad \text{or} \quad 2.
\end{aligned}
\tag{21}
$$

The shear forces $\tilde{Q}_{\gamma\zeta}$ appear in the second equation (21) if the independent shear parameters are $\hat{\dot{\omega}}_\beta = \hat{\dot{\gamma}}^0_{\gamma\beta} - \hat{\dot{w}}_{,\beta}$; then, it is sufficient to add $-\Sigma_{\gamma=1}^{2} \tilde{Q}_{\gamma\zeta} n_\gamma$ to the second member of the second equation in (21) and to replace $\bar{\Gamma}^{(w)}$ by $\bar{\Gamma}'^{(w)}$.

—from (9), instead of the natural boundary conditions (21), we can prescribe the displacement on the edge of the shell; these are essential boundary conditions.

Then, for example, to study the global free edge effects and traction edge effects we need natural boundary conditions (21). To take into account prescribed displacements, we must use essential boundary conditions. In eqns (20) and (21) we have noted from (14) and (11a):

$$
\hat{\mathscr{P}}_a = \int_\Omega \sum_{\beta=1}^{2} \rho(\ddot{U}_\beta \hat{\dot{U}}_\beta + \ddot{w}\hat{\dot{w}})\, dv = \int_{\mathscr{A}} \left(\sum_{\beta=1}^{2} \{\Gamma^{(u)}_\beta \hat{\dot{u}}_\beta + \Gamma^{(\gamma)}_\beta \hat{\dot{\gamma}}^0_{\gamma\beta}\} + \Gamma^{(w)} \hat{\dot{w}} \right) da - \int_{\mathscr{C}} \bar{\Gamma}^{(w)} \hat{\dot{w}}\, ds.
\tag{22}
$$

In addition, in (21) $\mathbf{m}$ is the tangent unit vector to $\mathscr{C}$ and τ is the curvilinear abscissa on $\mathscr{C}$.

Equations (20) and (21) are governing equations of the shell following lines of curvature coordinates.

Finally, eqns (20_1), (20_2), (21_1), (21_2), (21_3) are those of the classical shell theory; eqns (20_3) and (21_4) are due to the shear deformation.

4. CONSTITUTIVE LAW. EXACT SOLUTIONS FOR CROSS-PLY LAMINATED SHELLS

Equations (20) are valid for any anisotropic and linearly elastic materials. To have exact solutions, materials must be restricted to the orthotropic (Reddy, 1984b). Therefore, the constitutive law for the kth lamina is recognized as

$$
\sigma_{ij} = C_{ijkl} \varepsilon_{kl}; \quad \sigma_{\zeta\zeta} = 0.
\tag{23}
$$

In equation (23), because of the orthotropic material, if $ij = 11, 22, \zeta\zeta$ then $kl = 11, 22, \zeta\zeta$ (with summation), and if $ij = 2\zeta, 1\zeta, 12$ then kl is respectively equal to $2\zeta, 1\zeta, 12$ without summation. Other moduli C_{ijkl} are zero.

The hypothesis $\sigma_{\zeta\zeta} = 0$ (the normal transverse stress is neglected) is standard for moderately thick structures and is justified, Di Sciuva (1986). From (23) and the above remarks the local constitutive law becomes

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{2\zeta} \\ \sigma_{\zeta 1} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C'_{1111} & C'_{1122} & 0 & 0 & 0 \\ C'_{1122} & C'_{2222} & 0 & 0 & 0 \\ 0 & 0 & C_{2\zeta 2\zeta} & 0 & 0 \\ 0 & 0 & 0 & C_{\zeta 1\zeta 1} & 0 \\ 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{2\zeta} \\ 2\varepsilon_{\zeta 1} \\ 2\varepsilon_{12} \end{Bmatrix} \tag{24}$$

where $C'_{\alpha\alpha\beta\beta}$ takes into account

$$\sigma_{\zeta\zeta} = 0 : C'_{\alpha\alpha\beta\beta} = C_{\alpha\alpha\beta\beta} - C_{\alpha\alpha\zeta\zeta} C_{\zeta\zeta\beta\beta} / C_{\zeta\zeta\zeta\zeta}, \quad \text{no sum of } \alpha, \beta, \zeta.$$

In (24), the strains ε_{ij} are computed in the same way as the virtual strain rates, but by using the displacement field defined in (9), instead of the virtual velocity field. Then the strains can be deduced from (13) by exchange of $\hat{D}$ into ε and by omitting the hat from $\hat{\bar{u}}_\alpha$, $\hat{\bar{w}}$ and $\hat{\bar{\gamma}}^0_\alpha$.

From (24), of course, we have:

$$\sigma_{1\zeta}(\xi_1, \xi_2, \pm h/2, t) = \sigma_{2\zeta}(\xi_1, \xi_2, \pm h/2, t) = 0.$$

So, we can write the global constitutive law from (19), (24) and (13) (considering the above remarks). We obtain

—the global membrane constitutive law:

$$\mathbf{N} = \underset{\sim}{\mathbf{A}}\mathbf{V} - \underset{\sim}{\mathbf{B}}\mathbf{W} + \underset{\sim}{\tilde{\mathbf{B}}}\mathbf{Y}, \tag{25}$$

—the global first bending and twisting constitutive law:

$$\mathbf{M} = \mathbf{B}\mathbf{V} - \underset{\sim}{\mathbf{D}}\mathbf{W} + \mathbf{d}\mathbf{Y}, \tag{26}$$

—the global higher-order bending and twisting constitutive law:

$$\tilde{\mathbf{M}} = \underset{\sim}{\tilde{\mathbf{B}}}\mathbf{V} - \underset{\sim}{\mathbf{d}}\mathbf{W} + \underset{\sim}{\tilde{\mathbf{D}}}\mathbf{Y}, \tag{27}$$

—the global transverse shear constitutive law:

$$\tilde{\mathbf{Q}} = \underset{\sim}{\tilde{\mathbf{A}}}\mathbf{T}. \tag{28}$$

In these global constitutive laws, we have put

$$\begin{aligned}
\mathbf{N}^T &= \{N_{11}, N_{22}, N_{12}\}; \quad \mathbf{M}^T = \{M_{11}, M_{22}, M_{12}\} \\
\tilde{\mathbf{M}}^T &= \{\tilde{M}_{11}, \tilde{M}_{22}, \tilde{M}_{12}\}; \quad \tilde{\mathbf{Q}}^T = \{\tilde{Q}_{2\zeta}, \tilde{Q}_{1\zeta}\} \\
\mathbf{V}^T &= \left\{ \frac{\bar{u}_{1|1}}{\alpha_1} + \frac{\bar{w}}{R_1}, \ \frac{\bar{u}_{2|2}}{\alpha_2} + \frac{\bar{w}}{R_2}, \ \frac{\bar{u}_{1|2}}{\alpha_2} + \frac{\bar{u}_{2|1}}{\alpha_1} \right\} \\
\mathbf{W}^T &= \left\{ \frac{\bar{w}_{|11}}{\alpha_1^2}, \ \frac{\bar{w}_{|22}}{\alpha_2^2}, \ 2\frac{\bar{w}_{|12}}{\alpha_1 \alpha_2} \right\} \\
\mathbf{Y}^T &= \left\{ \frac{\bar{\gamma}^0_{1|1}}{\alpha_1}, \ \frac{\bar{\gamma}^0_{2|2}}{\alpha_2}, \ \frac{\bar{\gamma}^0_{1|2}}{\alpha_2} + \frac{\bar{\gamma}^0_{2|1}}{\alpha_1} \right\} \\
\mathbf{T}^T &= \{\bar{\gamma}^0_2, \bar{\gamma}^0_1\}.
\end{aligned} \tag{29}$$

These global constitutive equations explain the equilibrium eqns (20) for moderately thick laminated shells with orthotropic materials.

The symmetric matrices A, B, $\tilde{B}$, D, d, $\tilde{D}$ and $\tilde{A}$ are defined by ($\alpha\beta$ = 11, 22, 12, 66; $\phi\psi$ = 44, 55):

$$A_{\alpha\beta} = \int_{-h/2}^{+h/2} C'_{\alpha\beta}\,\mathrm{d}\zeta;\quad B_{\alpha\beta} = \int_{-h/2}^{+h/2} \zeta C'_{\alpha\beta}\,\mathrm{d}\zeta;\quad \tilde{B}_{\alpha\beta} = \int_{-h/2}^{+h/2} f(\zeta) C'_{\alpha\beta}\,\mathrm{d}\zeta$$

$$D_{\alpha\beta} = \int_{-h/2}^{+h/2} \zeta^2 C'_{\alpha\beta}\,\mathrm{d}\zeta;\quad d_{\alpha\beta} = \int_{-h/2}^{+h/2} \zeta f(\zeta) C'_{\alpha\beta}\,\mathrm{d}\zeta$$

$$\tilde{D}_{\alpha\beta} = \int_{-h/2}^{+h/2} f^2(\zeta) C'_{\alpha\beta}\,\mathrm{d}\zeta;\quad \tilde{A}_{\psi\phi} = \int_{-h/2}^{+h/2} f_{,\zeta}^2 C_{\psi\phi}\,\mathrm{d}\zeta. \tag{30}$$

The material coefficients $C'_{\alpha\beta}$ are deduced from (24) by the following convention:

$$C'_{\alpha\alpha\beta\beta} \to C'_{\alpha\beta};\quad C_{2323} \to C_{44};\quad C_{1313} \to C_{55} \quad \text{and} \quad C_{1212} \to C_{66} = C'_{66}.$$

From (30) and (25)–(29), we remark:

(1) the classical shell theory is deduced by taking $f(\zeta) = 0$,
(2) the first-order shear deformation shell theory is obtained when $f(\zeta) = \zeta$.

Exact solutions of the partial differential eqns (20) in arbitrary domains and for general boundary conditions is not possible. However, for simply-supported shells whose projection in the Cartesian x_1x_2-plane is a rectangle (spherically curved dome) we can solve these equations exactly, provided the lamination scheme is of antisymmetric cross-ply or symmetric cross-ply, orthotropic type. Exact solutions are also possible with cross-ply cylindrical shells as with cross-ply cylindrically curved panels. So, for a spherical dome, the boundary conditions are assumed to be of the form:

$$\begin{aligned}
\bar{u}_1(x_1, 0, t) &= \bar{u}_1(x_1, b, t) = \bar{u}_2(0, x_2, t) = \bar{u}_2(a, x_2, t) = 0\\
\bar{w}(x_1, 0, t) &= \bar{w}(x_1, b, t) = \bar{w}(0, x_2, t) = \bar{w}(a, x_2, t) = 0\\
N_{22}(x_1, 0, t) &= N_{22}(x_1, b, t) = N_{11}(0, x_2, t) = N_{11}(a, x_2, t) = 0\\
M_{22}(x_1, 0, t) &= M_{22}(x_1, b, t) = M_{11}(0, x_2, t) = M_{11}(a, x_2, t) = 0\\
\tilde{M}_{22}(x_1, 0, t) &= \tilde{M}_{22}(x_1, b, t) = \tilde{M}_{11}(0, x_2, t) = \tilde{M}_{11}(a, x_2, t) = 0\\
\bar{\gamma}_1^0(x_1, 0, t) &= \bar{\gamma}_1^0(x_1, b, t) = \bar{\gamma}_2^0(0, x_2, t) = \bar{\gamma}_2^0(a, x_2, t) = 0
\end{aligned} \tag{31}$$

where a and b denote the lengths along the x_1- and x_2-directions, respectively. In Cartesian coordinates x_i, we have from (4), $\mathrm{d}x_i = \alpha_i\,\mathrm{d}\xi_i$, $i = 1, 2$. Following the Navier solution procedure and for harmonic vibrations, we assume the following solution form that satisfies the boundary conditions in eqn (31) and the equilibrium equations in eqn (20) when the external forces p_α and couples m_α are zero:

$$\begin{aligned}
\bar{u}_1(\xi_1, \xi_2, t) &= \sum_{m,n=1}^{r} \bar{U}^1_{mn} \cos\lambda_m x_1 \sin\lambda_n x_2 \exp(\mathrm{i}\omega_{mn} t)\\
\bar{u}_2(\xi_1, \xi_2, t) &= \sum_{m,n=1}^{r} \bar{U}^2_{mn} \sin\lambda_m x_1 \cos\lambda_n x_2 \exp(\mathrm{i}\omega_{mn} t)\\
\bar{w}(\xi_1, \xi_2, t) &= \sum_{m,n=1}^{r} \bar{w}_{mn} \sin\lambda_m x_1 \sin\lambda_n x_2 \exp(\mathrm{i}\omega_{mn} t)\\
\bar{\gamma}_1^0(\xi_1, \xi_2, t) &= \sum_{m,n=1}^{r} \bar{\Gamma}^1_{mn} \cos\lambda_m x_1 \sin\lambda_n x_2 \exp(\mathrm{i}\omega_{mn} t)\\
\bar{\gamma}_2^0(\xi_1, \xi_2, t) &= \sum_{m,n=1}^{r} \bar{\Gamma}^2_{mn} \sin\lambda_m x_1 \cos\lambda_n x_2 \exp(\mathrm{i}\omega_{mn} t)
\end{aligned}$$

$$q(\xi_1, \xi_2, t) = \sum_{m,n=1}^{\infty} Q_{mn} \sin \lambda_m x_1 \sin \lambda_n x_2 \exp(i\omega_{mn} t) \tag{32}$$

where $\lambda_m = m\pi/a$, $\lambda_n = n\pi/b$ (m and n are integers), and where $\bar{U}^1_{mn}, \ldots, Q_{mn}$ are the amplitudes; ω_{mn} the frequency of harmonic vibrations. Substituting eqn (32) into eqn (20), we obtain an algebraic system in $\bar{U}^1_{mn}, \ldots, \bar{\Gamma}^2_{mn}$, for any integer m and n, Q_{mn} being a data:

$$(\underline{\mathbf{K}}^{(mn)} - \omega^2_{mn}\underline{\mathbf{M}}^{(mn)})\mathbf{U}_{mn} = \mathbf{Q}_{mn} \tag{33}$$

where

$$\mathbf{U}^{\mathrm{T}}_{mn} = \{\bar{U}^1_{mn}, \bar{U}^2_{mn}, \bar{W}_{mn}, \bar{\Gamma}^1_{mn}, \bar{\Gamma}^2_{mn}\}$$
$$\mathbf{Q}^{\mathrm{T}}_{mn} = \{0, 0, Q_{mn}, 0, 0\} \tag{34}$$

and where $\underline{\mathbf{K}}^{(mn)}$ and $\underline{\mathbf{M}}^{(mn)}$ are (5×5) squared matrix. The symmetric matrix $\underline{\mathbf{L}}^{(mn)} = \underline{\mathbf{K}}^{(mn)} - \omega^2_{mn}\underline{\mathbf{M}}^{(mn)}$ is given in the Appendix. Equation (33) can be solved for $\mathbf{U}_{mn}$, for each m and n.

The solution is then given by eqn (32), using a finite number of terms in the series: if the load q is a doubly sinusoidal load, then only $m = 1$ and $n = 1$, i.e. $\mathbf{U}_{11}$ is useful; if the load is uniform, then we must retain sufficient terms in (32) to obtain the convergence of the solution. For free vibration analysis, $Q_{mn} = 0$ and eqn (33) can be expressed as an eigenvalue equation in ω^2_{mn}. For static bending analysis, eqn (33) is solved with $\omega_{mn} = 0$.

For a cylindrical shell of x_1 axis, that is freely supported along its curved edges, only the boundary conditions based on $\bar{u}_2$, $\bar{w}(0, x_2, t)$, $\bar{w}(a, x_2, t)$, N_{11}, M_{11}, $\tilde{M}_{11}$, $\bar{\gamma}^0_2$ in eqn (31) need to be satisfied with (32) by taking $\lambda_m = m\pi/a$, $\lambda_n = n/r$ where R is the radius of the cylindrical shell, and so $R_1 = \infty$, $R_2 = R$. For a cylindrical panel simply supported on all its edges, eqns (31) and (32) are valid, and it is sufficient to take $R_1 = \infty$ and $R_2 = R$.

5. NUMERICAL RESULTS

The theory is evaluated through two sample problems. Numerical results from several other problems obtained by Pegoraro (1992) in statics and by Béakou (1991) in dynamics, confirm the trend observed from the following problems.

5.1. *Problem 1: Simply-supported cross-ply spherical dome under sinusoidal static load*

The material properties are: $E_L = 25\ E_T$, $G_{LT} = 0.5\ E_T$, $G_{TT} = 0.2\ E_T$, $\nu_{LT} = 0.25$, where E is the Young modulus, G the shear modulus, ν the Poisson ratio, L the longitudinal fiber direction and T the transverse fiber direction. Table 1 contains the non-dimensionalized center deflection of various cross-ply shells under sinusoidal static load. Three lamination schemes are tested: (0°/90°), (0°/90°/0°) and (0°/90°/90°/0°), to represent antisymmetric and symmetric cross-ply lamination schemes. Layers are of equal thickness.

For moderately thick spherical domes under a sinusoidal transverse static load and with a symmetric cross-ply lamination, the higher-order Reddy–Liu theory (Reddy and Liu 1985), underpredicts the central deflection when compared to the present theory (Table 1). We note the present theory is the nearest to the three-dimensional solution (Pagano, 1970) for plate under sinusoidal transverse load. For antisymmetric moderately thick cross-ply spherical domes, the trend reverses. Results from the shear deformation theory are obtained by Reddy (1984a,b) with a shear correction factor equal to 5/6. The maximum difference between the present analysis and the Reddy–Liu theory is of the order of one per cent, for moderately thick shells. For thin shells, quasi-identical results are obtained both for antisymmetric and symmetric laminations.

5.2. *Problem 2: Free vibrations of simply-supported short cylindrical isotropic and laminated shells*

Table 2 shows the nondimensionalized fundamental frequencies of cross-ply short cylindrical shells. The material properties and the lamination schemes are the same as those

Table 1. Non-dimensionalized center deflections, $\bar{w} = (-w(a/2, a/2)h^3E_T/Q_{11}a^4)10^3$ of simply-supported cross-ply laminated spherical domes under a sinusoidally distributed load ($a/h = 1$, $R_1 = R_2 = R$, $Q_{11} = 100$ SI). Each layer has an equal thickness. (SDT: shear deformation theory)

R/a	Theory	(0°/90°) $a/h = 100$	(0°/90°) $a/h = 10$	(0°/90°/0°) $a/h = 100$	(0°/90°/0°) $a/h = 10$	(0°/90°/90°/0°) $a/h = 100$	(0°/90°/90°/0°) $a/h = 10$
5	Reddy–Liu	1.1937	11.166	1.0321	6.7688	1.0264	6.7865
	Present	1.1937	11.142	1.0321	6.8168	1.0264	6.8321
	SDT	1.1948	11.429	1.0337	6.4253	1.0279	6.3623
10	Reddy–Liu	3.5733	11.896	2.4099	7.0325	2.4024	7.0536
	Present	3.5733	11.868	2.4101	7.0864	2.4026	7.1029
	SDT	3.5760	12.123	2.4109	6.6247	2.4030	6.5595
20	Reddy–Liu	7.1270	12.094	3.6170	7.1016	3.6133	7.1237
	Present	7.1235	12.065	3.6137	7.1566	3.6137	7.1740
	SDT	7.1270	12.309	3.6150	6.6756	3.6104	6.6099
50	Reddy–Liu	9.8692	12.150	4.2071	7.1212	4.2071	7.1436
	Present	9.8689	12.121	4.2077	7.1765	4.2077	7.1970
	SDT	9.8717	12.362	4.2027	6.6902	4.2015	6.6244
100	Reddy–Liu	10.444	12.158	4.3074	7.1240	4.3082	7.1464
	Present	10.444	12.129	4.3081	7.1794	4.3088	7.1970
	SDT	10.446	12.370	4.3026	6.6923	4.3021	6.6264
Plate	Reddy–Liu	10.651	12.161	4.3420	7.1250	4.3430	7.1474
	Present	10.651	12.132	4.3427	7.1803	4.3436	7.1980
	Exact-3D	—	—	4.3470	7.3700	4.3850	7.4300
	SDT	10.653	12.373	4.3370	6.6939	4.3368	6.6280

used in the problem discussed above. For moderately thick symmetric cross-ply cylindrical shells, the Reddy–Liu theory slightly over-estimates the fundamental natural frequencies when compared to the present theory. In the case of the moderately thick antisymmetric cross-ply cylindrical shells, it is the opposite. For a thin cross-ply shell, Reddy and Liu (1985) and the present results are identical. Results from the shear deformation theory are obtained by Reddy (1984a, b) with shear correction factors equal to 5/6.

Table 2. Non-dimensionalized fundamental frequencies of cross-ply cylindrical shells simply supported at the ends, $\bar{\omega} = (\omega a^2/h)\sqrt{\rho/E_T}$, a is the length of the shell, h its thickness. Each layer has an identical thickness. (SDT: shear deformation theory)

R/a	Theory	(0°/90°) $a/h = 100$	(0°/90°) $a/h = 10$	(0°/90°/0°) $a/h = 100$	(0°/90°/0°) $a/h = 10$	(0°/90°/90°/0°) $a/h = 100$	(0°/90°/90°/0°) $a/h = 10$
5	Reddy–Liu	16.690	9.0230	20.330	11.850	20.360	11.830
	Present	16.708	9.1060	20.333	11.800	20.366	11.791
	SDT	16.668	8.9082	20.332	12.207	20.361	12.267
10	Reddy–Liu	11.840	8.9790	16.620	11.800	16.630	11.790
	Present	11.848	9.0257	16.620	11.758	16.629	11.749
	SDT	11.831	8.8879	16.625	12.173	16.634	12.236
20	Reddy–Liu	10.270	8.9720	15.550	11.790	15.550	11.780
	Present	10.273	9.0011	15.550	11.748	15.550	11.739
	SDT	10.265	8.8900	15.556	12.166	15.559	12.230
50	Reddy–Liu	9.7830	8.9730	15.240	11.790	15.230	11.780
	Present	9.7850	8.9913	15.230	11.745	15.233	11.736
	SDT	9.7816	8.8951	15.244	12.163	15.245	12.228
100	Reddy–Liu	9.7120	8.9750	15.190	11.790	15.190	11.780
	Present	9.7130	8.9889	15.190	11.745	15.187	11.735
	SDT	9.7108	8.8974	15.198	12.163	15.199	12.227
Plate	Reddy–Liu	9.6880	8.9760	15.170	11.790	15.170	11.780
	Present	9.6883	8.9869	15.174	11.745	15.172	11.735
	SDT	9.6873	8.8998	15.183	12.162	15.184	12.226

Table 3. Free vibration analysis of simply-supported (at the ends) circular cylindrical isotropic shells. Comparison of lowest natural frequency parameters $\bar{\omega} = (\omega h/\pi)\sqrt{\rho/G}$ where G is the shear modulus, $\nu = 0.3$ the Poisson ratio and for $\lambda = \lambda_m R = m\pi R/a = 4\pi$, R is the mean radius of the cylinder, a its length and m an integer (SDT: shear deformation theory)

	$h/R = 0.06$				$h/R = 0.10$			
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
Exact-3D	0.08639	0.08748	0.08933	0.09199	0.20529	0.20802	0.21261	0.21906
Present	0.08635	0.08745	0.08931	0.09200	0.20458	0.20733	0.21192	0.21839
Bhimaraddi	0.08639	0.08728	0.08911	0.09175	0.20478	0.20678	0.21132	0.21771
SDT	0.08611	0.08718	0.08902	0.09165	0.20360	0.20628	0.21077	0.21710
Flügge	0.09161	0.09290	0.09510	0.09824	0.23623	0.23995	0.24620	0.25502
	$h/R = 0.12$				$h/R = 0.18$			
	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
Exact-3D	0.27491	0.27849	0.28447	0.29287	0.50338	0.50937	0.51934	0.53325
Present	0.27361	0.27721	0.28321	0.29161	0.50002	0.50606	0.51610	0.53008
Bhimaraddi	0.27286	0.27641	0.28233	0.29064	0.49818	0.50418	0.51416	0.52808
SDT	0.27197	0.27547	0.28131	0.28951	0.49479	0.50058	0.51021	0.52366
Flügge	0.32960	0.33479	0.34349	0.35571	0.67100	0.68056	0.69634	0.71803

Table 3 contains non-dimensionalized natural frequencies for isotropic short cylindrical shells using various theories: three-dimensional elasticity, Armenakas *et al.* (1969); present theory; Bhimaraddi theory, Bhimaraddi (1984); shear deformation theory with a shear correction factor equal to $\pi^2/12$, Mirsky and Hermann (1957); and Flügge theory, Flügge (1960). In Table 3 we have only retained the most significant problem from the Bhimaraddi paper (1984), i.e. $\lambda = m\pi R/a = R\lambda_m = 4\pi$, where R is the radius of the cylindrical shell and a the length. Comparisons of the above theories show that the maximum error in the present analysis is about -0.6%, in the Bhimaraddi results it is about -1%, in the shear deformation theory it is about -1.8%, and in the Flügge theory it is about $+35\%$; for $\lambda = 4\pi$, $n = 4$, $h/R = 0.18$. For lower values of λ the present theory also gives good results for the first circumferential mode ($n = 1$) and $\lambda = 0.5\pi$, π; and for $n = 1, 2, 3, 4$ when $\lambda = 2\pi$.

5.3. *Finite element approximations*

The kinematics defined by eqn (8) have been used to build a C^1 finite element for shallow shells which has been implemented in a standard computer code. From numerical tests, it seems necessary to carefully choose between $\bar{\omega}_\beta$ or $\bar{\gamma}^0_\beta = \bar{w}_{|\beta}/\alpha_\beta + \bar{\omega}_\beta$ in order to interpolate the shear in eqn (8). For shallow shells which are the aim of the present paper, the element stiffness matrix is computed using eqn (15), and eqns (25)–(30). So, from eqns (30), and since $f(\zeta) = h/\pi \sin(\pi\zeta/h)$, it is evident that *all integrals are analytically evaluated keeping the sine function intact* (i.e. without any polynomial approximations). Then, the sine model is always in competition with third-order shear deformation theories. The element has been found to be very accurate for evaluating stress distributions. An option for multilayered shallow shells allows having exact interface continuity between layers for displacements and transverse shear stresses (Béakou, 1991). In the midsurface, the classic numerical integration rules are used.

6. CONCLUDING REMARKS

A new refined shear deformation theory is presented for laminated shells, restricted to those for which thickness to radius of the curvature ratio is small compared to unity, in order to prove numerically *the efficiency of a new kinematics for shells* through standard problems. The reduction of the three-dimensional problem to a bidimensional one is accomplished assuming a displacement field containing a sine function associated to the shear. This displacement field has a three-dimensional justification for plates. The theory accounts for cosinusoidal variation of transverse shear strains according to thickness while no shear correction factors are needed, and for the same generalized displacements as in the shear-deformation theory, the boundary conditions upon the top and bottom surfaces of the shallow shell being exactly satisfied. A variationally consistent derivation of the boundary value problem (interior equations and edge conditions) is presented and numerical results for standard problems have shown the accuracy of the model.

The contact problem between layers of multilayered structures can be solved as, for instance, in Di Sciuva (1987). It is also possible to take into account the transverse normal stress for significant problems such as hygrothermal stresses in composite structures, sandwich structures of which the core rigidity is small compared to those of the skins. For this, to have only five generalized displacements, we suggest adding the $(h/\pi) f_{,3}\omega_3^0$ and $ff_{,3}\delta_3^0$ expressions to the $\bar{U}_3$ component of the displacement and to constrain functions ω_3^0 and δ_3^0 in order to satisfy the boundary conditions concerning the transverse normal stress.

The advantage of the present theory is in its simplicity, accuracy, no higher-order derivatives and material dependency of the kinematics which will allow extending the model to any nonlinear behaviour (geometric and material).

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APPENDIX: COMPONENTS OF THE SYMMETRIC MATRIX L

$$L_{11} = -A_{11}\lambda_m^2 - A_{66}\lambda_n^2 + \left(I_0 + \frac{2}{R_1}I_1 + \frac{1}{R_1^2}I_2\right)\omega^2, \quad L_{12} = -(A_{12}+A_{66})\lambda_m\lambda_n$$

$$L_{13} = \left(\frac{A_{11}}{R_1} + \frac{A_{12}}{R_2}\right)\lambda_m + B_{11}\lambda_m^3 + (B_{12}+2B_{66})\lambda_m\lambda_n^2 - \lambda_m\left(I_1 + \frac{1}{R_1}I_2\right)\omega^2$$

$$L_{14} = \tilde{B}_{11}\lambda_m^2 - \tilde{B}_{66}\lambda_n^2 + \left(J_1 + \frac{1}{R_1}K\right)\omega^2, \quad L_{15} = -(\tilde{B}_{12}+\tilde{B}_{66})\lambda_m\lambda_n$$

$$L_{22} = -A_{22}\lambda_n^2 - A_{66}\lambda_m^2 + \left(I_0 + \frac{2}{R_2}I_1 + \frac{1}{R_2^2}I_2\right)\omega^2$$

$$L_{23} = (B_{12}+2B_{66})\lambda_m^2\lambda_n + B_{22}\lambda_n^3 + \left(\frac{A_{22}}{R_2} + \frac{A_{12}}{R_1}\right)\lambda_n - \lambda_n\left(I_1 + \frac{1}{R_2}I_2\right)\omega^2$$

$$L_{24} = -(\tilde{B}_{12}+\tilde{B}_{66})\lambda_m\lambda_n, \quad L_{25} = -\tilde{B}_{22}\lambda_n^2 - \tilde{B}_{66}\lambda_m^2 + \left(J_1 + \frac{1}{R_2}K\right)\omega^2$$

$$L_{33} = -D_{11}\lambda_m^4 - D_{22}\lambda_n^4 - 2(D_{12}+2D_{66})\lambda_m^2\lambda_n^2 - 2\left(\frac{B_{11}}{R_1} + \frac{B_{12}}{R_2}\right)\lambda_m^2 - 2\left(\frac{B_{22}}{R_2} + \frac{B_{12}}{R_1}\right)\lambda_n^2$$
$$- \left(\frac{A_{11}}{R_1^2} + 2\frac{A_{12}}{R_1R_2} + \frac{A_{22}}{R_2^2}\right) + (\lambda_m^2 I_2 + \lambda_n^2 I_2 + I_0)\omega^2$$

$$L_{34} = d_{11}\lambda_m^3 + (d_{12}+2d_{66})\lambda_m\lambda_n^2 + \left(\frac{\tilde{B}_{11}}{R_1} + \frac{\tilde{B}_{12}}{R_2}\right)\lambda_m - \lambda_m K\omega^2$$

$$L_{35} = \mathrm{d}_{22}\lambda_n^3 + (d_{12}+2\mathrm{d}_{66})\lambda_m^2\lambda_n + \left(\frac{\tilde{B}_{22}}{R_2} + \frac{\tilde{B}_{12}}{R_1}\right)\lambda_n - \lambda_n K\omega^2$$

$$L_{44} = -\tilde{D}_{11}\lambda_m^2 - \tilde{D}_{66}\lambda_n^2 - \tilde{A}_{55} + J_2\omega^2, \quad L_{45} = -(\tilde{D}_{12}+\tilde{D}_{66})\lambda_m\lambda_n$$

$$L_{55} = -\tilde{D}_{22}\lambda_n^2 - \tilde{D}_{66}\lambda_m^2 - \tilde{A}_{44} + J_2\omega^2.$$